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Non-linear conductivity and magnetoplasma waves in compensated metals and semi-metals

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Abstract. The existence of non-linear magnetoplasma waves in compensated metals and semi-metals in the presence of a strong magnetic field is predicted. Non-linearity in the case considered is caused by the influence of the magnetic field of the wave on the dynamics of the electrons and holes. The conductivity tensor is calculated neglecting the spatial dispersion and is shown to be in the non-linear regime a differential—with respect to time—operator which is a manifestation of the temporal dispersion effects. The shape of the wave solution obtained is determined by two parameters: the amplitude $\mathcal{H}$ and the phase velocity V. When the amplitude is small and $V < V_A$, where V_A is the Alfvén velocity, the solution transforms into the well-known linear magnetoplasma wave. It is shown that, contrary to the linear case, the non-linear magnetoplasma wave exists when the phase velocity is both less and larger than V_A. It is established that with increase of the velocity and the amplitude being fixed the quasiharmonic wave turns into a series of pulses, the interval between which is growing infinitely. In the aperiodic limit the wave becomes a one-parameter soliton. Its velocity is larger than V_A and depends linearly on $\mathcal{H}$. With increase of $\mathcal{H}$, when V is fixed, the period of the magnetoplasma wave descends and the wave shape becomes a series of sharp spikes. Thus, when $V < V_A$ we have transition from a linear wave to an anharmonic one, while when $V > V_A$ we have a transition from a soliton to a sequence of pulses. Both the soliton and the non-linear periodic wave with $V > V_A$ have no analogues in the linear case. These electromagnetic waves are essentially non-linear even at small—in comparison with the external magnetic field—amplitudes.

1. Introduction

It is already known that metals possess rather peculiar non-linear electrodynamic properties [1]. Usually, in plasma or semiconductors, a non-linear response to electromagnetic perturbation is achieved owing to the considerable departure of the electron system from equilibrium. In metals, because of the high concentration, electrons are always in a near-equilibrium state. Nevertheless, it is fairly standard to observe a non-linear regime there, which is due to the fact that, in metals, sources of non-equilibrium and non-linearity are different. The former is caused by a weak electric field, while the latter is caused by a strong magnetic field of an electromagnetic wave. The Lorentz force, determined by the magnetic wave component, affects the dynamics of charge carriers. Hence, the conductivity of a metal depends on the configuration of the magnetic field. Such a magnetodynamic mechanism of non-linearity is typical for pure metals under low temperatures. It causes a wide range of non-linear electromagnetic phenomena.

In the static case, magnetodynamic non-linearity leads, e.g. to a deviation of the current–voltage characteristics of thin metal samples from Ohm's law [2, 3], to oscillations of voltage as a function of current [4, 5], to the appearance of a negative differential resistance [6] and to the pinch effect [7]. In the case in which the frequency ω of the electromagnetic field is much less than the carrier relaxation frequency ν, one should mention first the effect of the 'current states' [8–10], which, in its turn, causes a hysteresis interaction of radio-waves [11, 12] and the appearance of dissipative structures [13]. The effects of weak spatial dispersion in compensated metals lead to the formation of low-frequency electromagnetic solitons and kinks [14]. The action of a magnetic field on the dynamics of electrons causes a decrease of the collisionless damping of helicons [15]. That is why high-amplitude spiral waves can propagate under conditions in which linear electromagnetic excitations are absent [16].

At present, little is known about the manifestation of the magnetodynamic non-linearity in the high-frequency region ω,

$$\nu \ll \omega. \tag{1.1}$$

There has been some work on non-linear cyclotron resonance [17–19]. However, these studies were restricted to the case of weak non-linearity. At the same time, in the strongly non-linear regime in the high-frequency range (1.1), essentially new phenomena, having no analogues in the linear situation, may occur. In fact, the recent paper [20] predicted excitation of shock magnetoplasma waves in a compensated metal that is irradiated by a monochromatic signal in the presence of a constant magnetic field. It turns out that the magnetodynamic mechanism leads to steepening and overturning of the front of a wave propagating in the bulk of a sample. These results have stimulated the search for other non-trivial manifestations of the specifically metallic non-linearity in the high-frequency case (1.1).

In the present work we study the possibility of propagation of high-frequency finite-amplitude waves in compensated metals (or semi-metals) in the presence of a strong external magnetic field $\boldsymbol{H}_0$. This question is discussed for the case of an isotropic metal with quadratic electron and hole dispersion laws. The results obtained below for this simplified model will also be qualitatively valid for the more complicated structures of energy spectra of quasiparticles. We will restrict ourselves to the case of linearly polarized waves propagating transversely to the external magnetic field vector $\boldsymbol{H}_0$. The phase wave velocity V is assumed to be sufficiently large:

$$V/v_F^{e,h} \gg \min(1, \omega/\nu) \tag{1.2}$$

that we can neglect spatial dispersion effects in the conductivity. The expressions $v_F^{e,h}$ stand for the Fermi velocities of electrons v_F^e and holes v_F^h.

It is known [21] that in the linear case under conditions (1.1) and (1.2) in compensated metals (semi-metals) there exist transverse magnetoplasma waves. If the magnetic field of the wave is parallel to $\boldsymbol{H}_0$, its spectrum is determined by the conductivity in the plane perpendicular to $\boldsymbol{H}_0$. The transverse part of the conductivity tensor in the Cartesian coordinate frame x, y, $z \| \boldsymbol{H}_0$ is of the form [21]

$$\begin{aligned}
\sigma_{xx} = \sigma_{yy} &= \frac{Nec}{H_0}\left[\gamma_e\left(1+\gamma_e^2\right)^{-1} + \gamma_h\left(1+\gamma_h^2\right)^{-1}\right] \\
\sigma_{xy} = -\sigma_{yx} &= \frac{Nec}{H_0}\left[\left(1+\gamma_h^2\right)^{-1} - \left(1+\gamma_e^2\right)^{-1}\right]
\end{aligned} \tag{1.3}$$

where

$$\gamma_{e,h} = (\nu - \mathrm{i}\omega)\left(\frac{eH_0}{m_{e,h}c}\right)^{-1}. \tag{1.4}$$

Here c is the velocity of light, e is the elementary charge, the quantities m_e and m_h are effective electron and hole masses respectively and N is the concentration of either electrons or holes. Using (1.3) and (1.4), one can deduce the following dispersion relation between the frequency ω and wavevector $\boldsymbol{q}$ of the linear magnetoplasma wave [21] (for definiteness the vector $\boldsymbol{q}$ is chosen to be parallel to the x-axis):

$$q = \frac{\omega}{V_A}\left(1 - \frac{\omega^2}{\Omega_0^2}\right)^{-1/2} \qquad \Omega_0 = \frac{eH_0}{(m_e m_h)^{1/2} c}. \tag{1.5}$$

According to (1.5) the phase velocity $V = \omega/q$ of the harmonic wave is always less than the Alfvén velocity V_A. This means that the linear magnetoplasma waves can be observed only in such strong external fields H_0 that the Alfvén velocity V_A is much larger than the Fermi velocities, both electron, v_F^e, and hole, v_F^h:

$$v_F^{e,h} \ll V_A \qquad V_A = \frac{H_0}{[4\pi N(m_e + m_h)]^{1/2}}. \tag{1.6}$$

The results for the transverse magnetoplasma wave described above have been obtained in the limit of infinitesimal amplitudes, i.e. without taking into account magnetodynamic non-linearity. The question arises of what kind of magnetoplasma oscillation will take place in the non-linear regime, when the magnetic component affects charge motion and the fields in metals cease to be monochromatic. To study this problem we calculate in section 3 the response of a metal to a high-frequency electromagnetic field of finite amplitude. The expressions for the components of the non-linear conductivity turn out to be similar to the formulae (1.3) and (1.4). However, in the non-linear tensor, instead of the constant H_0, the sum of the external magnetic field and the intrinsic one appears. Moreover, in (1.4) the quantity $-\mathrm{i}\omega$ is replaced with the operator for differentiating with respect to time, $\partial/\partial t$. This takes into account temporal dispersion effects in non-linear, and hence non-monochromatic, wave fields. These distinctions lead to qualitatively new properties of σ_{xx}, σ_{yy}, σ_{xy}, σ_{yx}. They are not scalars any more, but differential operators acting on the electric field of the wave. This theory of the non-linear high-frequency conductivity enables one to perform self-consistent analysis of the dynamics of the transverse finite-amplitude magnetoplasma waves. In section 4 we will study waves of stationary shape which can propagate in compensated metals in the high-frequency case (1.1). Due to the magnetodynamic non-linearity, the domain of existence of physically interesting solutions of the Maxwell equations is enlarged. In subsection 4.2 we demonstrate the possibility of existence of non-linear magnetoplasma waves with a phase velocity larger than the Alfvén velocity, $V > V_A$. Such waves are periodic series of pulses, the interval between them increasing with increase of the velocity. Finally, the wave becomes a soliton propagating against a background of constant external field H_0. The soliton phase velocity depends linearly on its amplitude. It should be noted that the electromagnetic excitations discovered are essentially anharmonic even for small—in comparison with the external field—amplitudes. This means that even a relatively small degree of magnetodynamic non-linearity can be sufficient for the existence of strongly non-linear magnetoplasma waves.

2. Statement of the problem; general equations

In what follows we shall consider a one-dimensional problem, assuming that the electromagnetic field in metal depends only on one spatial coordinate, x. We assume that the external and intrinsic magnetic fields are collinear with each other and perpendicular to the direction of propagation (the x-axis). The z-axis is chosen to lie along the vector $\boldsymbol{H}(x,t)$ of the total magnetic field:

$$\boldsymbol{H}(x,t) = \{0, 0, H(x,t)\}. \tag{2.1}$$

The electric field vector, $\boldsymbol{E}(x,t)$, in this situation, is polarized in the (x, y)-plane:

$$\boldsymbol{E}(x,t) = \{E_x(x,t), E_y(x,t), 0\}. \tag{2.2}$$

Then, the Maxwell equations can be written as follows:

$$\frac{\partial H(x,t)}{\partial x} = -\frac{4\pi}{c} j_y(x,t) \qquad \frac{\partial E_y(x,t)}{\partial x} = -\frac{1}{c}\frac{\partial H(x,t)}{\partial t} \tag{2.3}$$

$$j_x(x,t) = 0 \tag{2.4}$$

where $j_x(x,t)$ and $j_y(x,t)$ are the x- and y-components of the current density. Equation (2.4) can be viewed as an electroneutrality condition. It can be used to determine the component transverse to the current, the so-called Hall component, $E_x(x,t)$, of the electric field.

To obtain a self-consistent solution of the set (2.3), (2.4) one has to evaluate the current-density components $j_{x,y}(x,t)$. Since the contributions of the electrons and holes are additive:

$$j_\alpha(x,t) = j_\alpha^e(x,t) + j_\alpha^h(x,t) \qquad \alpha = x, y \tag{2.5}$$

they can be calculated separately. The transverse components of the current density are given by the standard expression:

$$j_\alpha^i = -\frac{2e}{(2\pi\hbar)^3}\int \mathrm{d}^3\boldsymbol{p}\; v_\alpha^i f_i \qquad i = e, h. \tag{2.6}$$

Here $\hbar$ is the Planck constant, v_α^i is the α-component of the velocity and f_i is the non-equilibrium addition to the Fermi distribution function of the ith group of charge carriers. The integral in (2.6) is taken over the whole electron (hole) momentum space.

The non-equilibrium addition f_i should be determined from the Boltzmann kinetic equation:

$$\frac{\partial f_i}{\partial t} - \frac{e}{c}v_y^i H(x,t)\frac{\partial f_i}{\partial p_x} + \frac{e}{c}v_x^i H(x,t)\frac{\partial f_i}{\partial p_y} + \nu f_i = e\frac{\partial f_F}{\partial \varepsilon_i}\left[v_x^i E_x(x,t) + v_y^i E_y(x,t)\right] \tag{2.7}$$

where f_F is the equilibrium distribution function, which depends only on the electron (hole) energy ε_i. Equation (2.7) is linearized with respect to the electric field. The validity of such a linearization stems from the fact that in metals, due to the high conductivity, the electric field is always small. However, as was mentioned in the introduction, one may expect in metals an essentially non-linear regime because of the action of the intrinsic magnetic field of a wave on the dynamics of the charge carriers. That is why all of the terms in (2.7) related to the Lorentz force of the magnetic field $\boldsymbol{H}(x,t)$ have been retained. In the kinetic equation (2.7) we have taken into account the temporal dispersion and neglected the spatial dispersion. As is known, this may be done if the spatial scale of the electromagnetic field variation is the largest length of the problem, i.e. much larger than both the electron (hole) free path length and the typical size of their orbit in the magnetic field $\boldsymbol{H}(x,t)$. In our case these requirements for omitting the spatial dispersion can be written in the form of the inequality (1.2).

To conclude this section, it should be noted that we have assumed that the total magnetic field $H(x,t)$ never vanishes. This means that the function $H(x,t)$ is of constant sign. For definiteness we shall take both H_0 and $H(x,t)$ positive.

3. The non-linear current density and conductivity tensor

Let us begin by calculating the electron current density $j_\alpha^e(x,t)$. To this end, consider, instead of the non-equilibrium addition f_e and the velocities v_x^e, v_y^e, the function ψ_e and variables $v_\perp$, φ given by

$$f_e = e(\partial f_F/\partial\varepsilon_e)\psi_e \qquad v_x^e = -v_\perp\sin\varphi \qquad v_y^e = v_\perp\cos\varphi. \tag{3.1}$$

As follows from (2.7) and (3.1), the function ψ_e satisfies the equation

$$\frac{\partial\psi_e}{\partial t} + \Omega_e(x,t)\frac{\partial\psi_e}{\partial\varphi} + \nu\psi_e = v_\perp\left[-E_x\sin\varphi + E_y\cos\varphi\right] \tag{3.2}$$

where $\Omega_e(x,t)$ is the cyclotron electron frequency in the magnetic field $H(x,t)$:

$$\Omega_e(x,t) = \frac{eH(x,t)}{m_e c}. \tag{3.3}$$

Using the characteristics method one can obtain from (3.2) the following expression for ψ_e:

$$\begin{aligned}\psi_e(x,\varphi,t) = &-v_\perp\int_{-\infty}^{t}\mathrm{d}t'\,\sin\left(\varphi - \int_{t'}^{t}\mathrm{d}t''\,\Omega_e(x,t'')\right)\mathrm{e}^{\nu(t'-t)}E_x(x,t')\\ &+ v_\perp\int_{-\infty}^{t}\mathrm{d}t'\,\cos\left(\varphi - \int_{t'}^{t}\mathrm{d}t''\,\Omega_e(x,t'')\right)\mathrm{e}^{\nu(t'-t)}E_y(x,t').\end{aligned} \tag{3.4}$$

This formula determines rather complicated integral relations between the function ψ_e and electromagnetic field. This makes it inconvenient for further use. However, solution (3.4) can be rewritten in terms of differential, instead of integral, operators. Multiplying both sides of (3.4) by $\Omega_e(x,t)$ and acting with the operator $1+\hat\gamma_e^2$, where

$$\hat\gamma_e = \left(\nu + \frac{\partial}{\partial t}\right)\frac{1}{\Omega_e} \tag{3.5}$$

one gets the differential relation

$$\left(1+\hat\gamma_e^2\right)\Omega_e\psi_e = v_\perp\cos\varphi\left(E_x + \hat\gamma_e E_y\right) + v_\perp\sin\varphi\left(E_y - \hat\gamma_e E_x\right). \tag{3.6}$$

In line with this relation a solution of (3.2) can now be written as

$$\psi_e(x,\varphi,t) = \frac{v_\perp\cos\varphi}{\Omega_e}\left(1+\hat\gamma_e^2\right)^{-1}\left(E_x+\hat\gamma_e E_y\right) + \frac{v_\perp\sin\varphi}{\Omega_e}\left(1+\hat\gamma_e^2\right)^{-1}\left(E_y - \hat\gamma_e E_x\right). \tag{3.7}$$

We present in appendix A an alternative derivation of this result.

Using the expression obtained, equation (3.7), for $\psi_e(x,\varphi,t)$ one can easily calculate the electron current density. Substituting (3.1) and (3.7) in (2.6) and averaging over the momentum space, one can present $j_\alpha^e(x,t)$ as

$$\begin{aligned} j_x^e &= \frac{N_e ec}{H}\left(1+\hat\gamma_e^2\right)^{-1}\left(-E_y + \hat\gamma_e E_x\right)\\ j_y^e &= \frac{N_e ec}{H}\left(1+\hat\gamma_e^2\right)^{-1}\left(E_x + \hat\gamma_e E_y\right).\end{aligned} \tag{3.8}$$

Here N_e stands for the electron concentration.

The contribution of holes $j_\alpha^h(x,t)$ in the total current density (2.5) can be calculated analogously. Therefore, the corresponding formulae for $j_\alpha^h(x,t)$ follow from (3.8) on replacing the electron parameters with the hole ones:

$$e \to -e \qquad N_e \to N_h \qquad m_e \to m_h.$$

Also, instead of the operator $\hat{\gamma}_e$ there appears the differential operator $\hat{\gamma}_h$ and instead of $\Omega_e(x,t)$ there appears the hole cyclotron frequency $\Omega_h(x,t)$:

$$\hat{\gamma}_h = \left(\nu + \frac{\partial}{\partial t}\right)\frac{1}{\Omega_h} \qquad \Omega_h(x,t) = \frac{eH(x,t)}{m_h c}. \tag{3.9}$$

It should be noted that the operators $\hat{\gamma}_e$ and $\hat{\gamma}_h$ commute, since, as follows from (3.3), (3.5) and (3.9), they are linearly interrelated:

$$m_h\hat{\gamma}_e = m_e\hat{\gamma}_h. \tag{3.10}$$

As a result, the relation between the current density (2.5) and the electric field can be presented in the traditional form of Ohm's law:

$$j_\alpha = \hat{\sigma}_{\alpha\beta}E_\beta \qquad \alpha,\beta = x,y. \tag{3.11}$$

However, the components of the conductivity tensor $\hat{\sigma}_{\alpha\beta}$ in (3.11) are differential—with respect to time—operators:

$$\begin{aligned}\hat{\sigma}_{xx} = \hat{\sigma}_{yy} &= \frac{ec}{H}\left[N_e\hat{\gamma}_e\left(1+\hat{\gamma}_e^2\right)^{-1} + N_h\hat{\gamma}_h\left(1+\hat{\gamma}_h^2\right)^{-1}\right]\\ \hat{\sigma}_{xy} = -\hat{\sigma}_{yx} &= \frac{ec}{H}\left[N_h\left(1+\hat{\gamma}_h^2\right)^{-1} - N_e\left(1+\hat{\gamma}_e^2\right)^{-1}\right].\end{aligned} \tag{3.12}$$

We emphasize that formulae (3.11), (3.12) are valid only if the magnetic field $H(x,t)$ is strictly non-zero, which guarantees that the differential operators (3.5) and (3.9) have no singularities and that the above-developed procedure for reducing the integral equation (3.4) to the differential ones (3.7), (3.8) and (3.12) is correct.

Expressions (3.11), (3.12) constitute the principal result of this section. Formulae (3.12) determine the non-linear conductivity tensor for the situation in which charge carriers are affected by the magnetic field of the wave. Comparison of (3.12) with (1.3) demonstrates that the structure of the non-linear conductivity tensor is similar to the structure of the linear one, discussed in the introduction. That is why the linear limit of expressions (3.12) is readily obtainable. Indeed, let an electromagnetic excitation be sufficiently weak and depend on time according to the monochromatical law $\exp(-\mathrm{i}\omega t)$. In this case the charge-carrier movement is entirely determined by the constant external field H_0. Then one can put $H(x,t)$ in (3.12) equal to H_0 and replace the operators $\hat{\gamma}_{e,h}$ with the scalars $\gamma_{e,h}$. On doing this, for compensated metals ($N_e = N_h = N$), expressions (3.12) transform into the linear theory formulae (1.3).

For our further purposes it is convenient to rewrite (3.11) and (3.12) in a form containing no inverse operators $(1+\hat{\gamma}_{e,h}^2)^{-1}$. Multiplying relation (3.11) by H/ec and applying the operator $(1+\hat{\gamma}_e^2)(1+\hat{\gamma}_h^2)$ one can obtain for the x- and y-components of the current density the following expressions:

$$\begin{aligned}\left(1+\hat{\gamma}_e^2\right)\left(1+\hat{\gamma}_h^2\right)\frac{H}{ec}j_x &= \left[N_h\left(1+\hat{\gamma}_e^2\right) - N_e\left(1+\hat{\gamma}_h^2\right)\right]E_y\\ &\quad + \left[N_e\hat{\gamma}_e\left(1+\hat{\gamma}_h^2\right) + N_h\hat{\gamma}_h\left(1+\hat{\gamma}_e^2\right)\right]E_x\end{aligned} \tag{3.13}$$

$$\begin{aligned}\left(1+\hat{\gamma}_e^2\right)\left(1+\hat{\gamma}_h^2\right)\frac{H}{ec}j_y &= \left[N_e\left(1+\hat{\gamma}_h^2\right) - N_h\left(1+\hat{\gamma}_e^2\right)\right]E_x\\ &\quad + \left[N_e\hat{\gamma}_e\left(1+\hat{\gamma}_h^2\right) + N_h\hat{\gamma}_h\left(1+\hat{\gamma}_e^2\right)\right]E_y.\end{aligned} \tag{3.14}$$

According to the electroneutrality condition, $j_x = 0$, it follows from (3.13) that the transverse (i.e. the Hall) component E_x is related to the longitudinal component E_y via the differential relation

$$\left[N_e\hat{\gamma}_e\left(1+\hat{\gamma}_h^2\right) + N_h\hat{\gamma}_h\left(1+\hat{\gamma}_e^2\right)\right]E_x = \left[N_e\left(1+\hat{\gamma}_h^2\right) - N_h\left(1+\hat{\gamma}_e^2\right)\right]E_y. \tag{3.15}$$

Now we can exclude the Hall field from expression (3.14) for the y-component of the current density. Applying the operator $[N_e\hat{\gamma}_e(1+\hat{\gamma}_h^2)+N_h\hat{\gamma}_h(1+\hat{\gamma}_e^2)]$ to the both sides of equation (3.14) and using (3.15), one can derive after some simple calculations the following relation between the y-components of the current density and the electric field:

$$\left[N_e\hat{\gamma}_e\left(1+\hat{\gamma}_h^2\right)+N_h\hat{\gamma}_h\left(1+\hat{\gamma}_e^2\right)\right]\frac{H}{ec}\,j_y=\left[\left(N_e-N_h\right)^2+\left(N_e\hat{\gamma}_h+N_h\hat{\gamma}_e\right)^2\right]E_y. \tag{3.16}$$

The Maxwell equations (2.3), complemented with the equation (3.16), constitute a self-consistent system for determining the magnetic field and the y-components of the electric field in a metal. Knowing $H(x,t)$ and $E_y(x,t)$ one can then calculate the Hall field $E_x(x,t)$ using (3.15).

The above results are applicable for arbitrarily related concentrations of electrons N_e and holes N_h. In other words, they can be used for non-compensated metals as well as for compensated ones.

Let us discuss briefly the conductivity of non-compensated metals in the simplest case, in which there is only one (electron) group of charge carriers. Setting N_h in (3.16) equal to zero and taking into account the definition of the operator $\hat{\gamma}_e$, equations (3.3) and (3.5), one can conclude that the magnetodynamic non-linearity does not affect the longitudinal conductivity and that the y-component of the current density j_y does not depend explicitly on the magnetic field $H(x,t)$:

$$\left(\nu+\frac{\partial}{\partial t}\right)j_y=\frac{N_e e^2}{m_e}E_y. \tag{3.17}$$

This is so because of the strong non-linear Hall effect. Indeed, according to (3.15), the equation for determining the Hall electric field contains $H(x,t)$ explicitly:

$$\left(\nu+\frac{\partial}{\partial t}\right)\frac{E_x}{H}=\frac{e}{m_e c}E_y. \tag{3.18}$$

Consider now a compensated metal with $N_e=N_h=N$. In this case, taking into account expressions (3.3), (3.5), (3.9) for the operators $\hat{\gamma}_e$ and $\hat{\gamma}_h$, equation (3.15) for the Hall field $E_x(x,t)$ can be written as

$$E_x+\frac{m_e m_h c^2}{e^2}\left(\nu+\frac{\partial}{\partial t}\right)\left[\frac{1}{H}\left(\nu+\frac{\partial}{\partial t}\right)\frac{E_x}{H}\right]=\frac{(m_h-m_e)c}{e}\left(\nu+\frac{\partial}{\partial t}\right)\frac{E_y}{H}. \tag{3.19}$$

The right-hand side of this equation is non-zero due to the difference of the electron and hole masses. The relation between the longitudinal current density $j_y(x,t)$ and the electric field $E_y(x,t)$ turns out to be strongly non-linear with respect to the magnetic field $H(x,t)$:

$$Hj_y+\frac{m_e m_h c^2}{e^2}\left(\nu+\frac{\partial}{\partial t}\right)\left[\frac{1}{H}\left(\nu+\frac{\partial}{\partial t}\right)j_y\right]=N(m_e+m_h)c^2\left(\nu+\frac{\partial}{\partial t}\right)\frac{E_y}{H}. \tag{3.20}$$

In the linear limit (infinitesimal amplitudes, $H(x,t)\approx H_0$) formula (3.20), together with the Maxwell equations (2.3), in the high-frequency range (1.1), describes the propagation of a transverse monochromatic wave with the dispersion law (1.5). In the general case of finite amplitudes this formula can be used to study the transformation of magnetoplasma waves under the action of the magnetodynamic mechanism of non-linearity. The following section is devoted to this area.

4. Finite-amplitude magnetoplasma waves

4.1. Analytical solution of the problem

Consider now the problem of propagation of non-linear electromagnetic waves in a compensated metal in the case of high frequencies (1.1) and weak spatial dispersion (1.2). Omitting in (3.20) terms proportional to the relaxation frequency ν and eliminating therein the current density $j_y(x,t)$ via the first Maxwell equation (2.3) we come to the following set of equations for the electric and magnetic fields:

$$H\frac{\partial H}{\partial x}+\frac{m_e m_h c^2}{e^2}\frac{\partial}{\partial t}\left(\frac{1}{H}\frac{\partial^2 H}{\partial t\,\partial x}\right)=-4\pi N(m_e+m_h)c\frac{\partial}{\partial t}\left(\frac{E_y}{H}\right) \qquad (4.1)$$

$$\frac{\partial E_y}{\partial x}=-\frac{1}{c}\frac{\partial H}{\partial t}. \qquad (4.2)$$

We shall be looking for solutions of (4.1), (4.2) which have the form of stationary plane waves propagating in the x-direction:

$$E_y(x,t)=E_y(\tau) \qquad H(x,t)=H(\tau) \qquad \tau=t-x/V \qquad (4.3)$$

where τ is a new, running, variable. The partial differential equations (4.1), (4.2) become now ordinary differential equations with respect to τ. Integrating (4.2) one deduces the relation between the electric and magnetic fields:

$$E_y(\tau)=\frac{V}{c}\left[H(\tau)-H_0\right]. \qquad (4.4)$$

Here H_0 is an integration constant playing the role of an external magnetic field. Without loss of generality it can be chosen positive ($H_0>0$). Then the total magnetic field $H(\tau)$ should also be positive for all τs because from the very beginning $H(\tau)$ has been supposed to be of constant sign. So, we shall search for solutions of the set (4.1), (4.2) which satisfy the condition $H(\tau)>0$.

Now let us integrate equation (4.1) and exclude the electric field $E_y(\tau)$ using (4.4). After multiplying both sides of the resulting equation by $H\ \mathrm{d}H/\mathrm{d}\tau$ and integrating once more, one can obtain for the magnetic field $H(\tau)$ the following non-linear first-order differential equation:

$$-\left(\frac{2}{\Omega_0}\right)^2 H_0{}^2\left(\frac{\mathrm{d}}{\mathrm{d}\tau}H(\tau)\right)^2=P(H(\tau)). \qquad (4.5)$$

Here $P(H)$ is the fourth-order polynomial

$$P(H)=H^4+AH^2+8\left(\frac{V}{V_A}\right)^2 H_0{}^3 H+B \qquad (4.6)$$

with the coefficients A and B being arbitrary constants. This polynomial can be rewritten as

$$P(H)=(H-H_1)(H-H_2)(H-H_3)(H-H_4) \qquad (4.7)$$

where we use the designations H_1, H_2, H_3, H_4 for the roots of (4.6). Such a representation of $P(H)$, with the quantities H_1, H_2, H_3, H_4 playing the role of the parameters of the problem, instead of the coefficients A and B, is convenient for solving (4.5). We emphasize that $P(H)$ has no cubic term and that the coefficient of the linear term is positive. This imposes some restrictions on the roots of the polynomial (4.6):

$$H_1 + H_2 + H_3 + H_4 = 0 \tag{4.8}$$

$$8{H_0}^3\left(\frac{V}{V_A}\right)^2 = -(H_1 + H_2)(H_2 + H_3)(H_2 + H_4) > 0. \tag{4.9}$$

Hereafter we shall assume that at least two zeros of $P(H)$ are positive. This is necessary for the existence of a positive solution $H(\tau) > 0$ of the equation (4.5) (with τ varying, it will vary in the interval between these two positive roots). In this case the other two roots are real as follows from (4.8), (4.9). Moreover, according to (4.8) the least of them is necessarily negative. Without loss of generality we enumerate the zeros of $P(H)$ in descending order:

$$H_4 < H_3 < H_2 < H_1. \tag{4.10}$$

Then,

$$H_4 < 0 < H_2 < H_1 \qquad H_2 + H_4 < 0 < H_2 + H_3. \tag{4.11}$$

Since the left-hand side of (4.5) is negative, this equation possesses a real solution only when the polynomial $P(H)$ is also negative. Noting also that $P(H) \to +\infty$ as $H \to \pm\infty$ one can conclude that the positive and bounded solution $H(\tau) > 0$ of the differential equation (4.5) satisfies the inequalities

$$0 < H_2 \leqslant H(\tau) \leqslant H_1. \tag{4.12}$$

It is known [22, 23] that solutions of equations of the type (4.5) with fourth-order polynomials on the right-hand side can be presented as rational functions of squared elliptic sines. In our case, when all of the roots of the polynomial (4.7) are real, the solution $H(\tau)$ has the form

$$H(\tau) = \frac{H_1(H_2 - H_4) + H_4(H_1 - H_2)\,\mathrm{sn}^2(2K\tau/T;k)}{(H_2 - H_4) + (H_1 - H_2)\,\mathrm{sn}^2(2K\tau/T;k)}. \tag{4.13}$$

Here $K = K(k)$ is complete elliptic integral of the first kind. The modulus k of the elliptic sine can be expressed in terms of the roots of $P(H)$ as

$$k^2 = \frac{(H_1 - H_2)(H_3 - H_4)}{(H_1 - H_3)(H_2 - H_4)} \qquad 0 \leqslant k^2 \leqslant 1 \tag{4.14}$$

and, as follows from (4.10), it is a positive quantity less than unity. The squared elliptic sine $\mathrm{sn}^2(2K\tau/T;k)$ is periodic function of its argument $2K\tau/T$ with the period $2K$. Hence, the magnetic field of the wave $H(\tau)$ has the period T:

$$T = \frac{8K(k)}{\Omega_0}\frac{H_0}{[(H_1 - H_3)(H_2 - H_4)]^{1/2}}. \tag{4.15}$$

Since a squared elliptic sine varies from zero to unity, $H(\tau)$, according to (4.13), oscillates in the range (4.12) with the amplitude

$$\mathcal{H} = (H_1 - H_2)/2. \tag{4.16}$$

The phase velocity of the wave (4.13) is related to the roots of the polynomial (4.7) by the expression (4.9).

Let us separate in (4.13) the constant and alternating components of the magnetic field. To this end it is convenient to parametrize the roots of the polynomial (4.7) in terms of the elliptic functions $\mathrm{sn}(\alpha; k')$, $\mathrm{cn}(\alpha; k')$ and $\mathrm{dn}(\alpha; k')$ in the same way as was done in [24]:

$$\begin{aligned}
H_1 &= \frac{4H_0K(k)}{\Omega_0 T}\frac{1+\mathrm{dn}(\alpha;k')-\mathrm{cn}(\alpha;k')}{\mathrm{sn}(\alpha;k')}\\
H_2 &= \frac{4H_0K(k)}{\Omega_0 T}\frac{1-\mathrm{dn}(\alpha;k')+\mathrm{cn}(\alpha;k')}{\mathrm{sn}(\alpha;k')}\\
H_3 &= -\frac{4H_0K(k)}{\Omega_0 T}\frac{1-\mathrm{dn}(\alpha;k')-\mathrm{cn}(\alpha;k')}{\mathrm{sn}(\alpha;k')}\\
H_4 &= -\frac{4H_0K(k)}{\Omega_0 T}\frac{1+\mathrm{dn}(\alpha;k')+\mathrm{cn}(\alpha;k')}{\mathrm{sn}(\alpha;k')}.
\end{aligned} \tag{4.17}$$

Here the complementary modulus k' is defined by

$$k' = (1-k^2)^{1/2}. \tag{4.18}$$

It should be stressed that in terms of (4.17) the equation (4.8) for the roots of the polynomial (4.7), as well as the expressions (4.14) for the modulus k and (4.15) for the period T, become identities. In this way the equations (4.17) introduce new integration constants α, k and T instead of the old ones, the zeros of $P(H)$. Substituting (4.17) in (4.13) one can obtain after some calculations outlined in appendix B the following representation:

$$H(\tau) = H_0 + \mathrm{i}\frac{4H_0K}{\Omega_0 T}\left[Z\left(\frac{2K}{T}\tau - \frac{\mathrm{i}\alpha}{2};k\right) - Z\left(\frac{2K}{T}\tau + \frac{\mathrm{i}\alpha}{2};k\right)\right]. \tag{4.19}$$

Here Jacobi's function $Z(u;k)$ [22], which is the logarithmic derivative of one of the theta-functions, is a periodic function of its argument with the period $2K(k)$. Thus, the mean value over the period T of the second term in (4.19) is zero. The constant component of the magnetic field $H(\tau)$ is chosen equal to H_0, to ensure, together with (4.4), the vanishing of the mean value of the electric field. Such an additional requirement (the requirement of electroneutrality) leads to the wave period T no longer being an independent integration constant. It turns out to be related to the parameters α and k by

$$T(\alpha,k) = \frac{4K(k)}{\Omega_0}\left[\frac{\pi\alpha}{2K(k)K(k')} + \frac{\mathrm{cn}(\alpha;k')\,\mathrm{dn}(\alpha;k')}{\mathrm{sn}(\alpha;k')} + Z(\alpha;k')\right]. \tag{4.20}$$

The dependence of the phase velocity V and wave amplitude $\mathcal{H}$ on α and k follows from (4.9), (4.16) and (4.17):

$$V^2(\alpha,k) = {V_A}^2\left[\frac{4K(k)}{\Omega_0 T(\alpha,k)}\right]^3\frac{\mathrm{cn}(\alpha;k')\,\mathrm{dn}(\alpha;k')}{\mathrm{sn}^3(\alpha;k')} \tag{4.21}$$

$$\mathcal{H}(\alpha,k) = \frac{4H_0K(k)}{\Omega_0 T(\alpha,k)}\frac{\mathrm{dn}(\alpha;k')-\mathrm{cn}(\alpha;k')}{\mathrm{sn}(\alpha;k')}. \tag{4.22}$$

Thus, the expressions (4.19)–(4.22) determine the shape of the non-linear magnetoplasma wave, its period, velocity and amplitude. The region of admissible values of the independent parameters α and k is given by

$$0 \leqslant \alpha \leqslant K(k') \qquad 0 \leqslant k \leqslant 1. \tag{4.23}$$

The restrictions on the constant α stem from the conditions (4.10), (4.11) and (4.17), and the interval of values of k are rewritten from (4.14).

It is interesting that the expression (4.19) can be easily presented as a superposition of harmonic waves using the well-known Fourier series for the Jacobi Z-function (see [22]):

$$H(\tau) = H_0 + \frac{8\pi H_0}{\Omega_0 T} \sum_{n=1}^{\infty} \frac{\sinh(\pi n\alpha/2K(k))}{\sinh(\pi n K(k')/K(k))} \cos\left(\frac{2\pi n}{T}\tau\right). \tag{4.24}$$

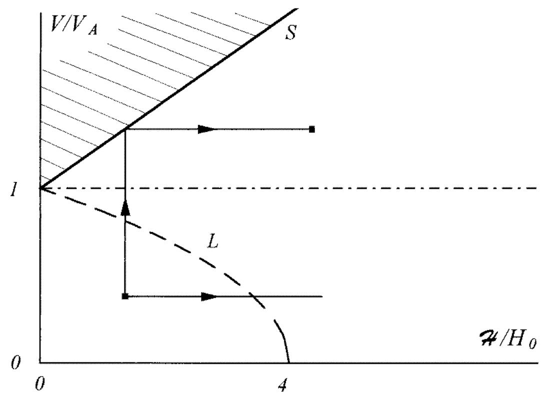

Figure 1. The range of admissible values of the amplitude and velocity of the magnetoplasma wave (the unshaded part of the $\mathcal{H}$–V plane). The line S corresponds to the soliton solution. The dashed curve L is the boundary of the quasilinear excitation region.

4.2. Analysis of the results

Let us study the form of the non-linear magnetoplasma wave, considering the amplitude $\mathcal{H}$ and the phase velocity V as independent parameters. In this case the quantities α, k and the period T are determined by the relations (4.20)–(4.22). The range of admissible values of $\mathcal{H}$ and V follows from the restrictions (4.23). It is depicted in figure 1 as the unshaded region of the $\mathcal{H}$–V plane.

The left-hand boundary of the $\mathcal{H}$–V diagram corresponds to the zero value of the modulus k:

$$\mathcal{H} = 0 \qquad 0 \leqslant V \leqslant V_A \qquad \text{at } k = 0. \tag{4.25}$$

This can be proved as follows. In accordance with (4.18), the complementary modulus $k' = 1$ when $k = 0$, and the elliptic functions become elementary ones [22]:

$$\begin{aligned} &K(k=0) = \pi/2 \qquad K(k' \to 1) = \ln(4/k) \to \infty \\ &\mathrm{sn}(\alpha; k'=1) = Z(\alpha; k'=1) = \tanh\alpha \\ &\mathrm{cn}(\alpha; k'=1) = \mathrm{dn}(\alpha; k'=1) = 1/\cosh\alpha \\ &\mathrm{dn}(\alpha; k' \to 1) - \mathrm{cn}(\alpha; k' \to 1) = (k^2/2)\cosh\alpha\tanh^2\alpha. \end{aligned} \tag{4.26}$$

Substituting (4.26) in (4.20)–(4.22), one can find that in the $k = 0$ limit the amplitude of the wave is zero, $\mathcal{H} = 0$, and its period T and the parameter α are related to V by

$$\Omega_0 T = 2\pi\left(1 - V^2/V_A^2\right)^{-1/2} \qquad \cosh\alpha = V_A/V. \tag{4.27}$$

At the same time, it follows from (4.23) for $k = 0$ that $0 \leqslant \alpha \leqslant \infty$. This implies that the phase velocity V varies from zero to V_A; see (4.25).

The upper boundary line S in figure 1 corresponds to the unit value of the modulus k. This straight line is given by the equation

$$V = V_A(1 + \mathcal{H}/H_0) \qquad \text{at } k = 1. \tag{4.28}$$

Here $k' = 0$, and the elliptic functions are described by the following asymptotic expressions [22]:

$$\begin{aligned} &K(k \to 1) = \ln(4/k') \to \infty && K(k' = 0) = \pi/2 \\ &\mathrm{cn}(\alpha; k' = 0) = \cos\alpha && \mathrm{dn}(\alpha; k' = 0) = 1 \\ &\mathrm{sn}(\alpha; k' = 0) = \sin\alpha && Z(\alpha; k' = 0) = 0. \end{aligned} \tag{4.29}$$

From relations (4.20)–(4.23) together with (4.29) one can find that the wave period T tends to infinity at $k' \to 0$, while α, which varies from zero to $\pi/2$, can be expressed in terms of V:

$$\Omega_0 T = 4\cot(\alpha)\,\ln(4/k') \to \infty \qquad \cos\alpha = V_A/V \qquad 0 \leqslant \alpha \leqslant \pi/2. \tag{4.30}$$

The wave amplitude and velocity, we recall, are related in this case by the equation of the line S, equation (4.28).

The lower boundary of the $\mathcal{H}$–V domain in figure 1 is the positive semi-axis of $\mathcal{H}$:

$$0 \leqslant \mathcal{H} \leqslant \infty \qquad V = 0 \qquad \text{at } \alpha = K(k') \qquad 0 \leqslant k \leqslant 1. \tag{4.31}$$

To confirm this, one has to set α in (4.20)–(4.22) equal to its maximal value $K(k')$. Then (see [22]),

$$\mathrm{cn}(K(k'); k') = Z(K(k'); k') = 0 \qquad \mathrm{dn}(K(k'); k') = k \qquad \mathrm{sn}(K(k'); k') = 1 \tag{4.32}$$

and one can write the velocity, period and parameter k in the form

$$V = 0 \qquad T = 2\pi/\Omega_0 \qquad kK(k) = \pi\mathcal{H}/2H_0. \tag{4.33}$$

As follows from (4.33) the amplitude $\mathcal{H}$ increases from 0 to $+\infty$ as k increases from zero to unity.

From the above analysis and figure 1 we see that, for any positive value of the amplitude $\mathcal{H}$, the phase velocity V can vary from zero to the maximal value (4.28):

$$0 \leqslant \mathcal{H} \leqslant \infty \qquad 0 \leqslant V \leqslant V_A(1 + \mathcal{H}/H_0). \tag{4.34}$$

This means that the velocity of the non-linear magnetoplasma wave (4.19) can be either less or larger than that of the Alfvén one, V_A. In other words, the domain of existence of the magnetoplasma waves is wider in the non-linear case than in the linear one, in which electromagnetic waves propagate only when $V < V_A$ (see (1.5) in the introduction).

Now we calculate the linear asymptote of the magnetic field (4.19), in which the first term of the sum over n in the Fourier series (4.24) dominates. Obviously, this case corresponds to sufficiently small values of the modulus k, which enables one to replace $K(k)$ and $K(k')$ in (4.24) with their limiting values (4.26), and to use for T and α the formulae (4.27). The amplitude of the first harmonic in (4.24) is given by

$$\frac{8\pi H_0}{\Omega_0 T}\frac{\sinh(\pi\alpha/2K(k))}{\sinh(\pi K(k')/K(k))} = H_0\frac{k^2 V_A}{2V}\left(1 - \frac{V^2}{V_A^2}\right) = \mathcal{H} \tag{4.35}$$

and is asymptotically equal to the wave amplitude (4.22). Thus, in the linear limit the magnetoplasma wave can be written as

$$H(x,t) = H_0 + \mathcal{H}\cos\left[\frac{2\pi}{T}\left(t - \frac{x}{V}\right)\right]. \tag{4.36}$$

The corresponding dispersion law (the relation between the period T and the velocity V) is described by the first formula in (4.27). When rewritten in terms of the frequency $\omega = 2\pi/T$ and wavevector $q = \omega/V$, it takes the usual form (1.5) [21]. The existence domain of the linear wave (4.36), (4.27) can be determined as follows. The amplitude of the second harmonic in (4.24) is small in comparison with the amplitude (4.35), if $k^2 V_A/8V \ll 1$. Expressing the parameter $k^2 V_A/8V$ in this inequality in terms of the amplitude $\mathcal{H}$ using (4.35) one can get the criterion for the linearity of the magnetoplasma wave:

$$\mathcal{H}/4H_0 \ll 1 - (V/V_A)^2. \tag{4.37}$$

Inequality (4.37) determines the region in the $\mathcal{H}$–V plane situated under the curve $V = V_A(1 - \mathcal{H}/4H_0)^{1/2}$, which is depicted in figure 1 with the dotted line.

The condition (4.37) is broken when the amplitude $\mathcal{H}$ increases, as well as when the velocity V increases. In both cases the higher harmonics begin to play a significant role in forming the wave shape.

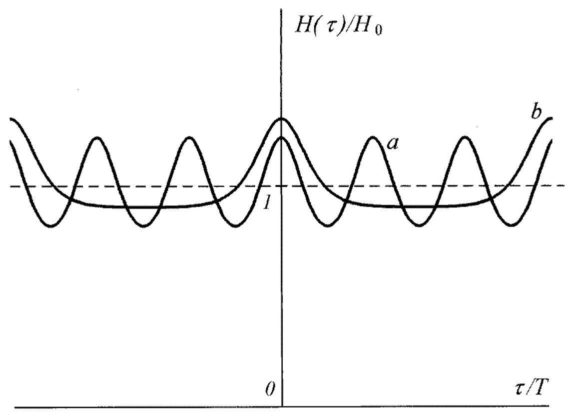

Figure 2. The shape of the magnetoplasma wave at the fixed amplitude $\mathcal{H} = 0.2H_0$ and for different values of the velocity: (a) $V = 0.7V_A$; (b) $V = 1.1V_A$.

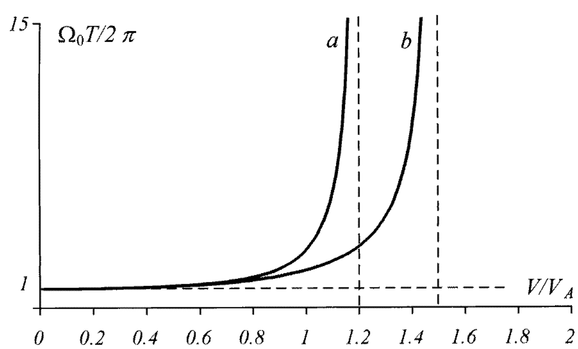

Figure 3. The dependence of the wave period on the velocity at the following fixed amplitudes: (a) $\mathcal{H} = 0.2H_0$; (b) $\mathcal{H} = 0.5H_0$.

In figure 2 the dependence of the magnetic field (4.24) on τ/T, when the amplitude is fixed, $\mathcal{H} = 0.2H_0$, is presented for two different values of the velocity V. Curve a corresponds to quasiharmonic oscillations and with good accuracy can be described by formula (4.36). Curve b represents the wave profile in the non-linear regime. One can see that non-linear oscillations $H(\tau/T)$ are less frequent than linear ones. The non-linear wave is a series of separated pulses, and its period $T(V/V_A)$ increases infinitely with growth of the velocity V (figure 3). Thereby $T(V/V_A)$ changes starting from its resonance value $T(0) = 2\pi/\Omega_0$ at $V = 0$ (see (4.33)). The aperiodic regime occurs when the phase velocity V takes a maximal—for a given amplitude $\mathcal{H}$—value (4.28). At this moment the point $(\mathcal{H}, V)$ reaches the line S in figure 1, and the magnetoplasma wave transforms into a solitary pulse, i.e. a soliton.

To obtain the analytical dependence of the magnetic field of the solitary wave on the variable τ we have to replace the Z-function in equation (4.19) with its asymptotic form, $Z(u; 1) = \tanh u$, and to use for the integral $K(k)$, the period T and the parameter α the formulae (4.29), (4.30). Some simple calculations lead to the following result:

$$H(x,t) = H_0 + 2H_0 \left\{ \frac{V^2}{V_A^2} - 1 \right\} \left\{ 1 + \frac{V}{V_A} \cosh \left[\Omega_0 \left(\frac{V^2}{V_A^2} - 1 \right)^{1/2} \left(t - \frac{x}{V} \right) \right] \right\}^{-1}. \quad (4.38)$$

From this expression one can see that the soliton propagates against the external magnetic field H_0. Its typical width $\Delta\tau$ is, in order of magnitude, equal to $2\pi\Omega_0^{-1}(V^2/V_A^2 - 1)^{-1/2}$. The phase velocity V is larger than the Alfvén one, V_A, and depends linearly on the amplitude $\mathcal{H}$ according to (4.28).

For growing $\mathcal{H}$ and fixed V one can observe transformations of the wave shape (4.24) which are inverse to the ones described above. As follows from figure 1 it is necessary to analyse two distinct cases: $V < V_A$ and $V > V_A$.

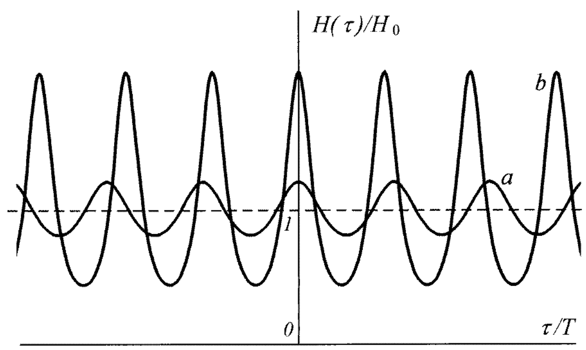

Figure 4. The shape of the magnetoplasma wave at the fixed velocity $V = 0.7V_A$ and different values of the amplitude: (a) $\mathcal{H} = 0.2H_0$; (b) $\mathcal{H} = 0.8H_0$.

Let us study first the case in which $V < V_A$ and the point $(\mathcal{H}, V)$ in figure 1 moves away from the quasilinear region, located under the dotted curve L. Figure 4 demonstrates a crossover from the harmonic wave (4.36) (curve a) to the non-linear one (curve b) for $V = 0.7V_A$. With $\mathcal{H}$ growing, the maxima of the function $H(\tau/T)$ become sharper, and the period $T(\mathcal{H}/H_0)$ decreases. When $\mathcal{H} = 0$, the period T has its largest, linear, value (4.27), while it tends to the constant $2\pi/\Omega_0$ with $\mathcal{H}/H_0$ going to infinity (see figure 5).

Consider now the magnetoplasma wave with a velocity larger than the Alfvén one, $V > V_A$. This case differs from the previous one, since the point $(\mathcal{H}, V)$ in figure 1 starts from the line S corresponding to the soliton (4.38), and does not leave the non-linear region. Such a wave for $V = 1.2V_A$ and two values of the amplitude $\mathcal{H}$ is depicted in figure 6.

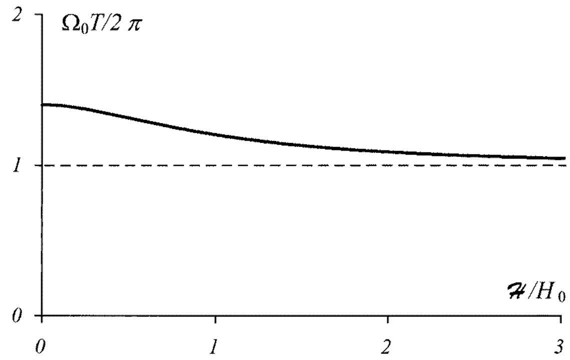

Figure 5. The dependence of the wave period on the amplitude at the fixed velocity $V = 0.7V_A$.

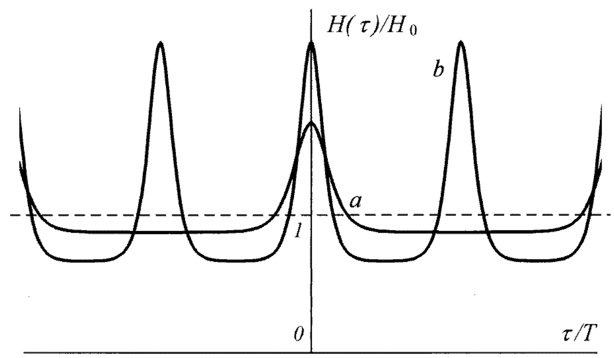

Figure 6. The shape of the magnetoplasma wave at the fixed velocity $V = 1.2V_A$ and different values of the amplitude: (a) $\mathcal{H} = 0.3H_0$; (b) $\mathcal{H} = 0.8H_0$.

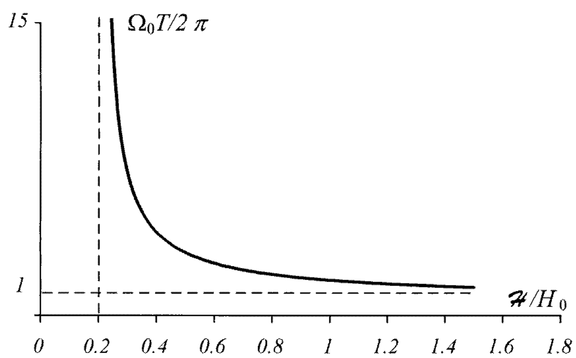

Figure 7. The dependence of the wave period on the amplitude at the fixed velocity $V = 1.2V_A$.

It is seen that for any $\mathcal{H}$ the magnetoplasma oscillations have the essentially non-linear form of isolated pulses. Their period $T(\mathcal{H}/H_0)$ is infinite when $\mathcal{H}$ has its minimal—for a given V—value, equal to the soliton amplitude (4.28). As $\mathcal{H}$ grows, the period T sharply decreases, tending to the constant $2\pi/\Omega_0$ (see figure 7).

Like the soliton, the periodic magnetoplasma wave with a velocity larger than V_A has no analogue in the linear case. It is formed exceptionally due to the self-action, caused by the magnetodynamic mechanism of non-linearity. Figure 6 convincingly demonstrates the fact that in the non-linear case the magnetoplasma wave possesses two characteristic scales of τ. The first one is the wave period T, while the second one is the time interval $\Delta\tau$, during which the wave field changes by a quantity of the order of the amplitude. In accordance with (4.19) the quantity $\Delta\tau$ is of the order of $\pi T/2K(k)$. Thus, in the non-linear situation the role of the frequency ω in the inequality (1.1) is played by $2\pi/\Delta\tau = 4K(k)/T$. Naturally, in the linear limit the two scales, T and $\Delta\tau$, coincide and $4K(k)/T$ becomes the frequency $2\pi/T$ of the harmonic wave (4.36).

In the analysis presented above it was assumed that the external magnetic field is non-zero, $H_0 \neq 0$. Now let us discuss the properties of the non-linear wave (4.19) in the absence of H_0. As follows from (4.34), for $H_0 = 0$ the domain of existence of the wave in the $\mathcal{H}$–V plane reduces to

$$0 \leqslant \mathcal{H} \leqslant \infty \qquad 0 \leqslant V \leqslant V_{\mathcal{H}} \qquad V_{\mathcal{H}} = \frac{\mathcal{H}}{[4\pi N(m_e + m_h)]^{1/2}}. \tag{4.39}$$

Thereby the quantity $V_{\mathcal{H}}$ has the meaning of the Alfvén velocity in the magnetic field equal to the amplitude $\mathcal{H}$. Formulae (4.20)–(4.22) yield that in the case where $H_0 \to 0$ the parameters α and k tend to $\pi/2$ and 1 respectively, and the wave period T approaches infinity. Hence, we can replace the Jacobi Z-function in (4.19) with its asymptotic form given by (4.26) and express the frequency scale $4K(k)/T$ in terms of the amplitude $\mathcal{H}$ using (4.22) and (4.29). After some simple transformations one finds that the wave (4.19) in the absence of an external field H_0 transforms into the soliton:

$$H(x,t) = 2\mathcal{H}\cosh^{-1}\left[\frac{e\mathcal{H}}{(m_e m_h)^{1/2}c}\left(t - \frac{x}{V}\right)\right]. \tag{4.40}$$

Its characteristic width, $\Delta\tau = 2\pi(m_e m_h)^{1/2}c/e\mathcal{H}$, is the geometric mean of the electron and hole cyclotron periods in the magnetic field equal to the amplitude $\mathcal{H}$ (compare with the expression (1.5) for Ω_0). When the phase velocity V takes its maximal—for a given $\mathcal{H}$—value $V = V_{\mathcal{H}}$, the two-parameter soliton (4.40) coincides with the one-parameter soliton (4.38) for $H_0 = 0$.

5. Conclusion

In the present work we have solved the problem of the non-linear magnetoplasma waves of finite amplitude in compensated metals. The mechanism of non-linearity is due to the self-action of the wave magnetic field. For this purpose we have calculated the conductivity tensor (3.12) in the absence of the spatial dispersion (1.2). In the non-linear regime its components are differential—with respect to time—operators, which is a manifestation of the temporal dispersion effects. It is shown that in non-compensated metal with a single group of charge carriers, non-linearity modifies the Hall effect considerably, but does not affect the longitudinal current density (see (3.18), (3.17)). At the same time, both the Hall and longitudinal fields turn out to be essentially non-linear in compensated metal (see (3.19), (3.20)).

We have obtained and analysed the analytical solution (4.4), (4.19)–(4.24) for the non-linear magnetoplasma wave in the case in which the total magnetic field is of constant sign. It has been shown that the dependence of the wave profile on the running variable $\tau = t - x/V$ is specified by two parameters, namely the amplitude $\mathcal{H}$ of the wave magnetic field and the phase velocity V. In comparison with the linear limit, when $V < V_A$, the range of admissible values of V in the non-linear regime is extended, so the phase velocity can be either less or larger than the Alfvén one, V_A.

We have demonstrated the transition of our solution to the well-known linear magnetoplasma wave (4.36), (4.27) [21] in the case of sufficiently small amplitudes $\mathcal{H}$ and $V < V_A$. Also, the range (4.37) of existence of the linear solution has been established. It was shown that for a fixed value of the amplitude $\mathcal{H}$ the primarily quasiharmonic wave profile transforms into a sequence of pulses, with the interval between them expanding infinitely, as the velocity V increases (see figures 2 and 3). When the phase velocity reaches the maximal possible value (4.28), the magnetoplasma wave becomes a soliton (4.38). According to the dispersion law (4.28), the soliton velocity V, being larger than the Alfvén one V_A, increases linearly with increase of the amplitude $\mathcal{H}$. Thus, the solitary wave (4.38) is a distinguished single-parameter solution. When the wave amplitude increases with the velocity kept fixed, the magnetoplasma oscillations take the form of sharp spikes. Their period decreases monotonically (see figures 5 and 7). In such a manner, the linear oscillations transform to anharmonic ones when $V < V_A$ (figure 4), while at $V > V_A$ the soliton turns into a periodic magnetoplasma wave (figure 6).

We pay special attention to the fact that the strongly non-linear regime can occur even in the case in which the wave amplitude is small in comparison with the external magnetic field. This enables experimental observation of the predicted non-linear excitations to be achieved. Indeed, the non-linear waves with small amplitudes have a phase velocity V close to V_A (see figure 1). In such a case, the condition (1.2), which allows one to neglect the spatial dispersion effects, coincides with the inequality (1.6). Therefore the non-linear excitations can be observed in the same range of the external magnetic fields H_0 as the linear waves exist in. In the bismuth-like semi-metals the Alfvén velocity V_A considerably exceeds the Fermi velocity $v_F^{e,h}$, if the external magnetic field H_0 is of the order of few thousands of oersteds (or higher). At the same time, the amplitudes of the alternating signal in today's experiments attain values of tens or hundreds of oersteds (see, e.g. [1, 12]). For just this reason, it is important that the non-linear effects discussed in our paper take place also in the case where $\mathcal{H} \ll H_0$.

In the studies presented above we restricted ourselves to the non-dissipative situation. However, in real conditions, even in the high-frequency region (1.1), wave propagation is always accompanied with weak damping due to collisions of electrons with scatterers. So, the question of the influence of a small relaxation frequency ν on the formation and propagation of the non-linear excitations discovered naturally arises. The investigation of the stability of the waves found is an important aspect as well, because some types of travelling wave are unstable against small perturbations (see, e.g. [25]). Besides, in both non-linear and linear cases there is the problem of taking into account the spatial dispersion effects, which is especially important when the phase velocity V is small and when the inequality (1.2) is violated. Finally, a remaining question is that of how to generate the predicted electromagnetic structures by means of an external signal. Such problems need special investigations and will be considered in future studies.

Appendix A. An alternative derivation of expression (3.7)

The aim of this section is to eliminate the integral operators in the expression (3.4) for the function $\psi_e(x, \varphi, t)$, which can be done using integration by parts. Consider the first term in (3.4):

$$\psi_1 = -v_\perp \int_{-\infty}^{t} dt'\ \sin[\varphi - \Phi(x, t', t)] e^{\nu(t'-t)} E_x\big(x, t'\big). \tag{A.1}$$

The function $\Phi(x, t', t)$ introduced here is defined by

$$\Phi(x, t', t) = \int_{t'}^{t} dt''\ \Omega_e(x, t''). \tag{A.2}$$

It is easy to see that in the integrand in (A.1) the sine can be presented as a derivative:

$$\sin[\varphi - \Phi(x, t', t)] = -\frac{1}{\Omega_e(x, t')} \frac{\partial}{\partial t'} \cos[\varphi - \Phi(x, t', t)]. \tag{A.3}$$

Substituting (A.3) in (A.1) and integrating by parts one can obtain

$$\psi_1 = \frac{v_\perp \cos\varphi}{\Omega_e} E_x - v_\perp \int_{-\infty}^{t} dt'\ \cos[\varphi - \Phi(x, t', t)] e^{\nu(t'-t)} \left[\left(\nu + \frac{\partial}{\partial t}\right) \frac{E_x}{\Omega_e}\right]_{t=t'}. \tag{A.4}$$

Now, presenting the function $\cos[\varphi - \Phi(x, t', t)]$ as a full derivative:

$$\cos[\varphi - \Phi(x, t', t)] = \frac{1}{\Omega_e(x, t')} \frac{\partial}{\partial t'} \sin[\varphi - \Phi(x, t', t)] \tag{A.5}$$

one can integrate (A.4) by parts. Then, using again (A.3) and integrating by parts once more one can write $\psi_1(x, \varphi, t)$ as

$$\begin{aligned}\psi_1 = {} & \frac{v_\perp \cos\varphi}{\Omega_e} \left\{E_x - \left(\nu + \frac{\partial}{\partial t}\right) \frac{1}{\Omega_e} \left(\nu + \frac{\partial}{\partial t}\right) \frac{E_x}{\Omega_e}\right\} - \frac{v_\perp \sin\varphi}{\Omega_e} \left(\nu + \frac{\partial}{\partial t}\right) \frac{E_x}{\Omega_e} \\ & + v_\perp \int_{-\infty}^{t} dt'\ \cos[\varphi - \Phi(x, t', t)] e^{\nu(t'-t)} \\ & \times \left[\left(\nu + \frac{\partial}{\partial t}\right) \frac{1}{\Omega_e} \left(\nu + \frac{\partial}{\partial t}\right) \frac{1}{\Omega_e} \left(\nu + \frac{\partial}{\partial t}\right) \frac{E_x}{\Omega_e}\right]_{t=t'}. \end{aligned} \tag{A.6}$$

As a result of the integration, the operator $\hat{\gamma}_e$, equation (3.5), arises in (A.6). Thus, integration by parts leads to an expansion of the integral operator in (A.1) in a power series in the differential operator $\hat{\gamma}_e$. Continuing the integration by parts in (A.6) one can get

$$\psi_1 = \frac{v_\perp \cos\varphi}{\Omega_e} \left[1 - \hat{\gamma}_e^2 + \hat{\gamma}_e^4 - \hat{\gamma}_e^6 + \cdots\right] E_x - \frac{v_\perp \sin\varphi}{\Omega_e} \hat{\gamma}_e \left[1 - \hat{\gamma}_e^2 + \hat{\gamma}_e^4 - \hat{\gamma}_e^6 + \cdots\right] E_x. \tag{A.7}$$

The sum in square brackets in (A.7) is the operator $(1 + \hat{\gamma}_e^2)^{-1}$:

$$\big(1 + \hat{\gamma}_e^2\big)^{-1} = \sum_{n=0}^{\infty} (-1)^n \hat{\gamma}_e^{2n}. \tag{A.8}$$

In this way, the final formula for the function $\psi_1(x, \varphi, t)$ takes the form

$$\psi_1 = \frac{v_\perp \cos\varphi}{\Omega_e} \big(1 + \hat{\gamma}_e^2\big)^{-1} E_x - \frac{v_\perp \sin\varphi}{\Omega_e} \hat{\gamma}_e \big(1 + \hat{\gamma}_e^2\big)^{-1} E_x. \tag{A.9}$$

The second integral in equation (3.4) can be presented analogously, which leads to the expression (3.7) for the function $\psi_e(x, \varphi, t)$.

Appendix B. Derivation of the representation (4.19) for the magnetic field $H(\tau)$

Substituting representation (4.17) for the roots of the polynomial $P(H)$ in the solution (4.13) we obtain

$$H(\tau) = \frac{4H_0K}{\Omega_0 T}\,\frac{1+\mathrm{dn}(\alpha;k')-\mathrm{cn}(\alpha;k')}{\mathrm{sn}(\alpha;k')} - \frac{4H_0K}{\Omega_0 T}\,\frac{2(\mathrm{dn}(\alpha;k')-\mathrm{cn}(\alpha;k'))(1+\mathrm{dn}(\alpha;k'))\,\mathrm{sn}^2(2K\tau/T;k)}{\mathrm{sn}(\alpha;k')[1+\mathrm{cn}(\alpha;k')+(\mathrm{dn}(\alpha;k')-\mathrm{cn}(\alpha;k'))\,\mathrm{sn}^2(2K\tau/T;k)]}. \tag{B.1}$$

Using the standard identities from [22] we express the functions $\mathrm{cn}(\alpha;k')$, $\mathrm{sn}(\alpha;k')$ and $\mathrm{dn}(\alpha;k')$ in the second summand of (B.1) via functions of $\mathrm{i}\alpha/2$ and rewrite $H(\tau)$ as

$$H(\tau) = \frac{4H_0K}{\Omega_0 T}\,\frac{1+\mathrm{dn}(\alpha;k')-\mathrm{cn}(\alpha;k')}{\mathrm{sn}(\alpha;k')} + \mathrm{i}\frac{4H_0K}{\Omega_0 T}\,\frac{2k^2\,\mathrm{sn}(\mathrm{i}\alpha/2;k)\,\mathrm{cn}(\mathrm{i}\alpha/2;k)\,\mathrm{dn}(\mathrm{i}\alpha/2;k)\,\mathrm{sn}^2(2K\tau/T;k)}{1-k^2\,\mathrm{sn}^2(\mathrm{i}\alpha/2;k)\,\mathrm{sn}^2(2K\tau/T;k)}. \tag{B.2}$$

Applying then the addition theorem for Jacobi's Z-functions [24]:

$$\frac{2k^2\,\mathrm{sn}(\mathrm{i}\alpha/2;k)\,\mathrm{cn}(\mathrm{i}\alpha/2;k)\,\mathrm{dn}(\mathrm{i}\alpha/2;k)\,\mathrm{sn}^2(2K\tau/T;k)}{1-k^2\,\mathrm{sn}^2(\mathrm{i}\alpha/2;k)\,\mathrm{sn}^2(2K\tau/T;k)} = Z\big(2K\tau/T-\mathrm{i}\alpha/2;k\big) - Z\big(2K\tau/T+\mathrm{i}\alpha/2;k\big) + 2Z\big(\mathrm{i}\alpha/2;k\big) \tag{B.3}$$

we substitute (B.3) in (B.2) and go over from the function $Z(\mathrm{i}\alpha/2;k)$ to the function $Z(\alpha;k')$, as is described in [22]. After some simple transformations, expression (B.2) for the magnetic field can be presented as

$$H(\tau) = \frac{4H_0K}{\Omega_0 T}\left[\frac{\pi\alpha}{2KK'} + \frac{\mathrm{cn}(\alpha;k')\,\mathrm{dn}(\alpha;k')}{\mathrm{sn}(\alpha;k')} + Z(\alpha;k')\right] + \mathrm{i}\frac{4H_0K}{\Omega_0 T}\left[Z\left(\frac{2K}{T}\tau-\frac{\mathrm{i}\alpha}{2};k\right) - Z\left(\frac{2K}{T}\tau+\frac{\mathrm{i}\alpha}{2};k\right)\right]. \tag{B.4}$$

Here K' stands for $K(k')$.

The electroneutrality condition can be formulated as the requirement for the mean value of $E_y(\tau)$ over the period T to be zero. As follows from (4.4), this is the case if the constant component of the magnetic field coincides with H_0. Setting the first term in (B.4) equal to H_0, we obtain the expressions (4.19) for the magnetic field $H(\tau)$ and (4.20) for the wave period T.

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